On Inexact Newton Methods for Inverse Problems in Banach Spaces

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Outline

1. Inverse Problems

2. Geometry of Banach Spaces

3. REGINN
    - Convergence Analysis
    - Examples
    - Numerical Experiments

Ill-Posed Problems

Inverse Problems

Find the CAUSE of a phenomenon from partial knowledge of the EFFECT produced by it.

Mathematically: Find x satisfying

$$F(x) = y,$$

$F : X \supset D(F) \to Y$ operates between *Banach* spaces.

Definition (Well-posedness (Hadamard))

1. Existence
2. Uniqueness
3. Stability

Regularization Methods

Noisy data: Find x satisfying

$$F(x) = y,$$

having

$$\|y^\delta - y\| \leq \delta.$$

(Regularization Property)

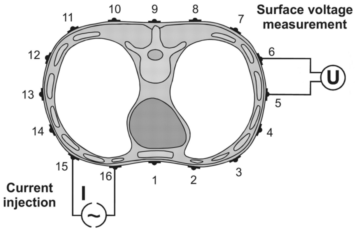
For each pair (y^δ, δ) find a vector

$$x_\delta \approx x^+$$

such that

$$x_\delta \to x^+ \qquad as \qquad \delta \to 0.$$

Example: Electrical Impedance Tomography



[I. Frerichs et al. 2001]

Neumann-to-Dirichlet operator

$$\Lambda_\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \qquad I \mapsto U$$

Inverse Problem: Find $\sigma \in L^\infty(\Omega)$ satisfying

$$F(\sigma) := \Lambda_\sigma,$$

having

$$\|\Lambda_\sigma^\delta - \Lambda_\sigma\|_{\mathcal{L}(\mathbb{R}^d,\mathbb{R}^d)} \leq \delta.$$

Difficulties in Banach Spaces

$$As = b, \qquad A : X \longrightarrow Y \text{ linear}$$

Hilbert spaces:

$\varphi(s) := \frac{1}{2}\|As - b\|^2$ is differentiable,

$\psi_k := \nabla\varphi(s_k) = A^*(As_k - b) \in X$,

$s_{k+1} := s_k - \lambda\psi_k$, with $\lambda > 0$.

Banach spaces:

$\varphi(s) := \frac{1}{r}\|As - b\|^r$, $r > 1$, is subdifferentiable,

$\psi_k \in \partial\varphi(s_k) \subset X^*$,

$$j_p : X \longrightarrow X^* \quad \text{or} \quad j_q^* : X^* \longrightarrow X \quad \text{is necessary.}$$

Duality Mapping

$$f(x) := \frac{1}{p}\|x\|^p, \qquad p > 1$$

Convexity: related to convexity of f

Smoothness: related to differentiability of f

Definition (Duality Mapping)

$$J_p(x) := \partial f(x)$$

Selection: $j_p : X \longrightarrow X^*, \qquad j_p(x) \in J_p(x)$

$$\langle j_p(x), y\rangle \leq \|x\|^{p-1}\|y\| \qquad \text{and} \qquad \langle j_p(x), x\rangle = \|x\|^p$$

Bregman Distance

Polarization Identity:

$$\frac{1}{2}\|x-y\|^2 = \frac{1}{2}\|x\|^2 - \langle y, x\rangle + \frac{1}{2}\|y\|^2$$

Definition (Bregman Distance)

$$\Delta_p(x,y) := \frac{1}{p}\|x\|^p - \langle j_p(y), x\rangle + \frac{1}{p^*}\|j_p(y)\|^{p^*}$$

Δ_p is not a metric

Definition

$$X \text{ is } p\text{-convex} \Leftrightarrow \Delta_p(x,y) \gtrsim \frac{1}{p}\|x-y\|^p \text{ for all } x,y \in X$$

$$X \text{ is } p\text{-smooth} \Leftrightarrow \Delta_p(x,y) \lesssim \frac{1}{p}\|x-y\|^p \text{ for all } x,y \in X$$

Smoothness/Convexity of Power Type

$$X \ p\text{−smooth} \Rightarrow X \text{ uniformly smooth} \Rightarrow X \text{ smooth}$$

$$X \ p\text{−convex} \Rightarrow X \text{ uniformly convex} \Rightarrow X \text{ strictly convex}$$

Examples

1. The spaces $L^p(\Omega)$, $W^{n,p}(\Omega)$ and $\ell^p(\mathbb{R})$, $1 < p < \infty$, are

$$\max\{p, 2\}\text{−convex} \quad \text{and} \quad \min\{p, 2\}\text{−smooth}$$

2. Hilbert spaces are 2−convex and 2−smooth

3. $J_p : L^p(\Omega) \longrightarrow L^{p^*}(\Omega), \qquad 1 < p < \infty$

$$J_p(g) = |g|^{p-1}\text{sgn}(g)$$

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REGularization INexact Newton

Algorithm (REGINN (A. Rieder, 1999))

Input: $x_0, F, y^\delta, \delta > 0, \mu \in (0,1), \tau > 1$

$n = 0$,

Repeat n

$s_{n,0} = 0, \quad k = 0, \quad A_n = F'(x_n), \quad b_n^\delta = y^\delta - F(x_n),$

Repeat k

Compute $s_{n,k+1} = h(s_{n,k})$ *from* $A_n s = b_n^\delta$,

Until $\left\| A_n s_{n,k} - b_n^\delta \right\| < \mu \left\| b_n^\delta \right\|$

$x_{n+1} = x_n + s_{n,k}$,

Until $\left\| b_n^\delta \right\| \leq \tau\delta$

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Convergence Analysis

Tangential **C**one **C**ondition: for all $x, y \in B \subset D(F)$

$$\|F(x) - F(y) - F'(y)(x-y)\| \leq \eta \|F(x) - F(y)\|, \qquad 0 \leq \eta < 1$$

(Assumption 1)

$$\lim_{k\to\infty} \|A_n s_{n,k} - b_n^\delta\| = \inf_{s\in X} \|A_n s - b_n^\delta\|$$

1. Inner iteration terminates
2. All (outer) iterates are well-defined
3. REGINN terminates after $N(\delta) \in \mathbb{N}$ iterations
4. $\|b_{n+1}^\delta\| \leq \Lambda \|b_n^\delta\|, \qquad 0 < \Lambda < 1$

Convergence Analysis

(Assumption 2)

$e_n := x^+ - x_n, \qquad and \qquad (\lambda_{n,k}) \subset \mathbb{R}_+$

$$\Delta_p(x^+, x_n + s_{n,k+1}) - \Delta_p(x^+, x_n + s_{n,k}) \leq \lambda_{n,k} \|A_n s_{n,k} - b_n^\delta\|^{r-1} \left(\|A_n e_n - b_n^\delta\| - C \|A_n s_{n,k} - b_n^\delta\| \right)$$

5. $\Delta_p(x^+, x_{n+1}) < \Delta_p(x^+, x_n)$

6. $(x_n)_n$ uniformly bounded on n and δ

7. $x_{N(\delta)} \rightharpoonup x^+$ as $\delta \longrightarrow 0$

Convergence Analysis

(Assumption 3)

$s_{n,0} = 0$ *and*

$$j_p(x_n + s_{n,k+1}) = j_p(x_n + s_{n,k}) - \lambda_{n,k} A_n^* j_r(v_{n,k}),$$

with $\|v_{n,k}\| \leq \|A_n s_{n,k} - b_n^\delta\|$

8. $x_n \longrightarrow x^+$ as $n \longrightarrow \infty$ $(\delta = 0)$

(Assumption 4)

$$s_{n,k}^\delta \longrightarrow s_{n,k}^0 \quad \textit{as} \quad \delta \longrightarrow 0$$

9. $x_{N(\delta)} \longrightarrow x^+$ as $\delta \longrightarrow 0$

A Concrete Example (Gradient Methods)

Hilbert spaces:

$\varphi_n(s) := \frac{1}{2}\|A_n s - b_n^\delta\|^2$ is differentiable,

$\psi_{n,k} := \nabla\varphi_n(s_{n,k}) = A_n^*(A_n s_{n,k} - b_n^\delta) \in X$,

$s_{n,k+1} := s_{n,k} - \lambda\psi_{n,k}$, with $\lambda > 0$.

Banach spaces:

$\varphi_n(s) := \frac{1}{r}\|A_n s - b_n^\delta\|^r,\ \ r > 1$, is subdifferentiable,

$\psi_{n,k} \in \partial\varphi_n(s_{n,k}) = A_n^* J_r(A_n s_{n,k} - b_n^\delta) \subset X^*$,

$j_p(x_n + s_{n,k+1}) := j_p(x_n + s_{n,k}) - \lambda\psi_{n,k}$.

Further Examples

Some methods satisfying Assumptions A1 to A4

1. Gradient methods
Landweber[1], Modified Steepest Descent, Minimal Error

2. Tikhonov methods
Iterated-Tikhonov[2], Tikhonov-Phillips

3. Mixed Gradient-Tikhonov methods

4. Kaczmarz methods[1,2]

1 [F.M., A. Rieder, A. Leitão. 2014], 2 [F.M., A. Rieder. 2014]

Electrical Impedance Tomography

Neumann-to-Dirichlet (or Current-to-Voltage) operator

$$\Lambda_\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \qquad I \mapsto U$$

Inverse Problem: Find $\sigma \in L^\infty(\Omega)$ satisfying

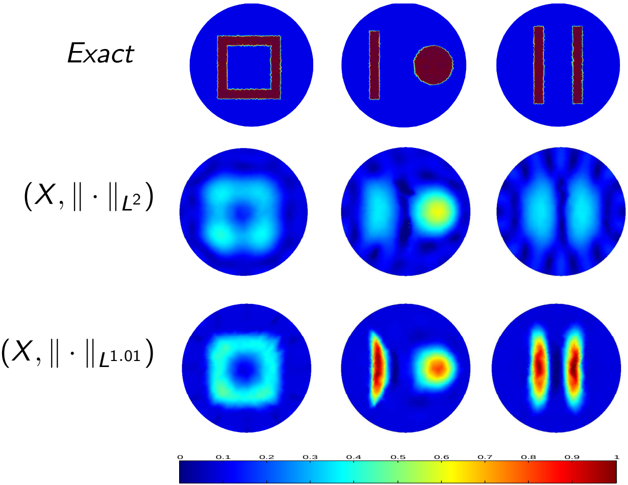
$$F(\sigma) := \Lambda_\sigma,$$

having

$$\|\Lambda_\sigma^\delta - \Lambda_\sigma\| \leq \delta$$

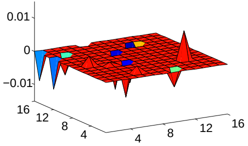
Sparsity Constraints

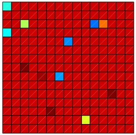
Mixed method. $\delta = 0.1\%$ Gaussian noise



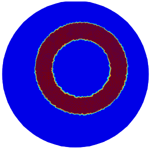
Impulsive Noise

Decreasing Error method. $\delta = 1\%$ impulsive noise

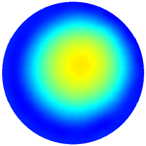




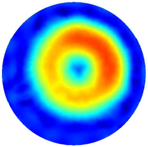
Exact

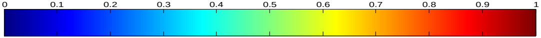


$(Y, \|\cdot\|_2)$



$(Y, \|\cdot\|_{1.01})$





Conclusions

1. General convergence analysis
2. Many different methods can be used
3. Good results for EIT

THANK YOU FOR YOUR ATTENTION!