A Projection Strategy for Choosing the Regularization Parameter of Iterated Tikhonov Method in Banach Spaces

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New Trends in Parameter Identification for Mathematical Models

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Regularization Methods

Noisy data: Find x satisfying

$$Ax = y,$$

having

$$\|y^\delta - y\| \leq \delta.$$

(Regularization Property)

For each pair (y^δ, δ) find a vector

$$x_\delta \approx x^+$$

such that

$$x_\delta \to x^+ \qquad as \qquad \delta \to 0.$$

Iterated Tikhonov

$$T_k(x) = \frac{1}{r}\|Ax - y^\delta\|^r + \alpha_k \Delta(x, x_{k-1}), \qquad \alpha_k > 0$$

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Iterated Tikhonov

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How to choose (α_k) ?

Common (a priori) choices:

- $\alpha_k = \text{constant}$
- $\alpha_k = r\alpha_{k-1}$, with $0 < r < 1$

Outline

Motivation in Hilbert Spaces

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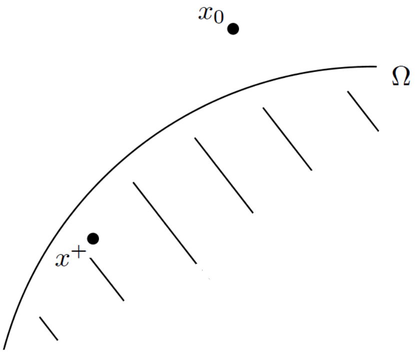
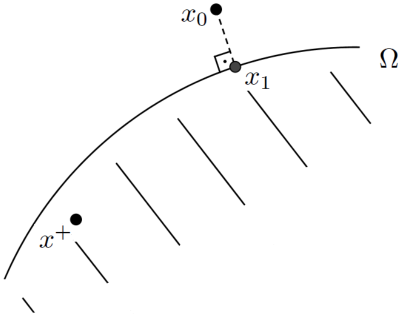


Ω closed and convex

$x^+ \in \Omega,\ x_0 \notin \Omega$

Motivation in Hilbert Spaces

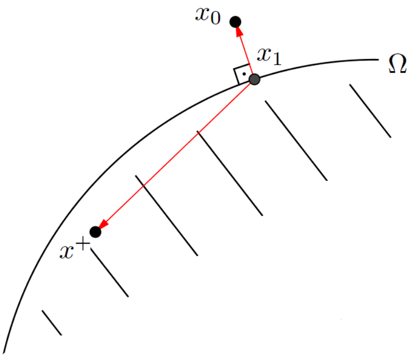


$$x_1 = P_\Omega(x_0)$$

Polarization identity:

$$\frac{1}{2}\left\|x^+ - x_1\right\|^2 - \frac{1}{2}\left\|x^+ - x_0\right\|^2 = -\frac{1}{2}\left\|x_1 - x_0\right\|^2 + \langle x_1 - x_0, x_1 - x^+\rangle$$

Motivation in Hilbert Spaces

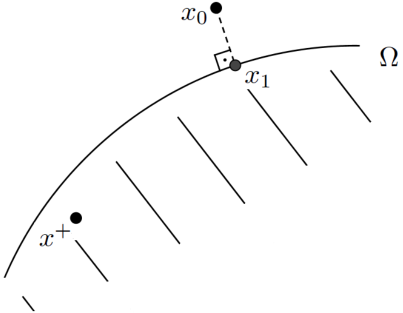


$$\langle x_1 - x_0, x_1 - x^+ \rangle \leq 0$$

Thus:

$$\frac{1}{2}\left\|x^+ - x_1\right\|^2 \leq \frac{1}{2}\left\|x^+ - x_0\right\|^2$$

Motivation in Hilbert Spaces



$$x_1 = P_\Omega(x_0)$$

$$\Leftrightarrow$$

x_1 solves

$$\begin{cases} \min \frac{1}{2} \|x - x_0\|^2 \\ \text{s.t.} \quad x \in \Omega \end{cases}$$

Bregman Distance

Subdifferential: $\varphi : X \longrightarrow \bar{\mathbb{R}}$ convex, X Banach, $x_0 \in X$

$$\partial\varphi(x_0) := \{\xi_0 \in X^* : \varphi(x) \geq \varphi(x_0) + \langle \xi_0, x - x_0 \rangle, \forall x \in X\}$$

Definition (Bregman Distance)

$\varphi : X \longrightarrow \bar{\mathbb{R}}$ convex, X Banach, $\xi_0 \in \partial\varphi(x_0)$

$$\Delta_{\xi_0}\varphi(x, x_0) := \varphi(x) - \varphi(x_0) - \langle \xi_0, x - x_0 \rangle$$

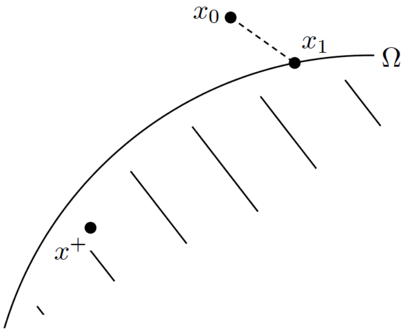
X Hilbert and $\varphi(x) = \frac{1}{2}\|x\|^2$ implies $\Delta_{\xi_0}\varphi(x, x_0) = \frac{1}{2}\|x - x_0\|^2$.

Three Points Identity:

$$\Delta_{\xi_1}\varphi(x^*, x_1) - \Delta_{\xi_0}\varphi(x^*, x_0) = -\Delta_{\xi_0}\varphi(x_1, x_0) + \langle \xi_1 - \xi_0, x_1 - x^* \rangle$$

for all $x^*, x_0, x_1 \in X, \xi_0 \in \partial\varphi(x_0)$ and $\xi_1 \in \partial\varphi(x_1)$.

Banach Spaces



Ω closed and convex

x_1 solves

$$\begin{cases} \min \Delta_{\xi_0}\varphi(x, x_0) \\ \text{s.t.} \quad x \in \Omega \end{cases}$$

Back to Inverse Problems

$$Ax = y, \qquad \|y - y^\delta\| \leq \delta$$

Define:

$$\Omega_\mu := \{x \in X : \|Ax - y^\delta\| \leq \mu\}, \qquad \mu > 0$$

- $\mu \geq \delta \Rightarrow x^+ \in \Omega_\mu$
- $\mu < \|Ax_0 - y^\delta\| \Rightarrow x_0 \notin \Omega_\mu$

Thus $\delta \leq \mu < \|Ax_0 - y^\delta\|$ implies Ω_μ separates x_0 and x^+.

Optimization Problem

$$\begin{cases} \min & \Delta_{\xi_0}\varphi(x, x_0) \\ \text{s.t.} & \|Ax - y^\delta\| \leq \mu \end{cases} \tag{1}$$

$$x_\lambda := \mathit{argmin}\left\{ \frac{\lambda}{r}\|Ax - y^\delta\|^r + \Delta_{\xi_0}\varphi(x, x_0) \right\}, \quad G(\lambda) := \|Ax_\lambda - y^\delta\| \tag{2}$$

Lemma

Assume $\delta < \mu < \|Ax_0 - y^\delta\|$. Are equivalent:

1. *x is a solution of (1);*
2. *$x = x_{\lambda^*}, \lambda^* > 0$ and $G(\lambda^*) = \mu$ in (2).*

Define

$$\mu := \eta\|Ax_0 - y^\delta\| + (1 - \eta)\delta, \qquad 0 < \eta_{\min} \leq \eta \leq \eta_{\max} < 1$$

Algorithm (Nonstationary Iterated Tikhonov)

[1] *choose an initial guess* $x_0 \in X$ *and* $\xi_0 \in \partial\varphi(x_0)$*;*

[2] *choose* $0 < \eta_{\min} \leq \eta_{\max} < 1$, $\tau > 1$ *and set* $k := 0$*;*

[3] *while* $\left(\|Ax_k - y^\delta\| > \tau\delta\right)$ *do*

[3.1] $k := k + 1$*;*

[3.2] *compute* λ_k, x_k *such that*

$$x_k = \arg\min \frac{\lambda_k}{r}\|Ax - y^\delta\|^r + \Delta_{\xi_{k-1}}\varphi(x, x_{k-1}) \text{ and}$$

$$\eta_{\min}\|Ax_{k-1} - y^\delta\| + (1-\eta_{\min})\delta \leq \|Ax_k - y^\delta\| \leq \eta_{\max}\|Ax_{k-1} - y^\delta\| + (1-\eta_{\max})\delta;$$

set $\xi_k = \xi_{k-1} - \lambda_k A^* J_r(Ax_k - y^\delta)$.

Main Results

(Assumptions)

a) $A : X \longrightarrow Y$ *is continuous;*

b) $y \in R(A)$*;*

c) φ *is weakly l.s.c. and uniformly convex;*

d) X *is reflexive and* Y *is uniformly smooth.*

1. Algorithm is well-defined and terminates;

2. $k_\delta \leq \log_{\eta_{\max}} \left(\frac{(\tau-1)\delta}{\|Ax_0 - y^\delta\| - \delta} \right) + 1$;

3. $\Delta_{\xi_k}\varphi(x^*, x_k) - \Delta_{\xi_{k-1}}\varphi(x^*, x_{k-1}) < -\lambda_k \left(1 - \frac{1}{\tau}\right) \|Ax_k - y^\delta\|^r$;

Main Results

4. There exists a unique solution $x^\dagger \in X$ satisfying

$$\Delta_{\xi_0}\varphi(x^\dagger, x_0) = \inf\{\Delta_{\xi_0}\varphi(x, x_0) : x \in D(\varphi), Ax = y\};$$

5. (noise-free convergence): $\delta = 0 \Rightarrow x_k \longrightarrow x^\dagger$ as $k \longrightarrow \infty$;

6. (regularization): $x_{k_\delta} \longrightarrow x^\dagger$ as $\delta \longrightarrow 0$.

References

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Thank you for your attention!